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On the paramagnetism of spin in the classical limit

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Abstract. Spin-$\frac{1}{2}$ particles subject to external fields will often exhibit paramagnetism. This effect can be formalised in terms of an inequality involving appropriate partition functions constructed with spin Hamiltonians which we choose to be of Pauli type. Whereas in quantum mechanics it is known that a paramagnetic inequality does not hold true in general, herein we investigate the paramagnetism of spin in the classical limit $\hbar \to 0$. We apply previous results to derive a classical partition function which, in satisfying the relevant inequality, gives rise to general paramagnetism.

1. Introduction

In quantum mechanics the notion of spin is introduced as an internal degree of freedom of certain quantum particles. The dynamical role of the spin shows up in its relation to the statistics of the attached particles. While such a relation has to be postulated for non-relativistic quantum mechanics, within relativistic quantum field theory it is proved and known as the celebrated spin-statistics theorem (Fierz 1939, Pauli 1940). Another aspect of the dynamical influence of spin is related to the behaviour of such particles in external magnetic fields. In general, paramagnetic properties of matter systems are attributed to the presence of particles with spin. It has been conjectured (Hogreve *et al* 1978) that the paramagnetic influence of spin is a universal law, but a counterexample (Avron and Simon 1979) has shown that this is not the case. Nevertheless, below we shall succeed in proving that at least in a certain classical limit the spin shows a general paramagnetic behaviour.

To be more precise, consider the partition function $Z(t; H)$, $t > 0$, corresponding to the quantum Hamiltonian H

$$Z(t; H) = \mathrm{Tr}\exp(-tH).$$

The Hamiltonians relevant for our analysis are of Pauli type

$$H_A = \left(\sum_{k=1}^{3} \sigma_k(-\mathrm{i}\hbar\nabla_k + A_k)\right)^2 + V. \tag{1}$$

Here σ_k, $k = 1, 2, 3$, are the familiar Pauli matrices and A_k are the components of the vector potential of an external electromagnetic field. Furthermore, we assume that the

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scalar potential V is such that $Z(t;\, H_A) < \infty$ for all $t > 0$ (precise conditions on V and the other involved quantities will be stated in § 2).

Because of the anticommutativity properties $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ of the Pauli matrices H_A can be equivalently expressed as

$$H_A = \sum_{k=1}^{3} (-\mathrm{i}\hbar\nabla_k + A_k)^2 - \tfrac{1}{2}\mathrm{i}\hbar \sum_{k,l=1}^{3} F_{kl}\sigma_{kl} + V \tag{2}$$

where $\sigma_{kl} = \frac{1}{2}(\sigma_k\sigma_l - \sigma_l\sigma_k)$ and F_{kl} is the field tensor $F_{kl} = \partial_k A_l - \partial_l A_k$. For identically vanishing vector potentials $A_k \equiv 0$ and fields $F_{kl} \equiv 0$ the corresponding Hamiltonian will be denoted by H_0.

Now a system is said to behave paramagnetically (diamagnetically) if by placing it in a magnetic field its free energy decreases (increases). This is equivalent to the increase (decrease) of the partition function if an external magnetic field is turned on. Hence a universal paramagnetism of spin is equivalent to the inequality

$$Z(t;\, H_A) \geqslant Z(t;\, H_0) \tag{3}$$

for all A and for all $t > 0$. Relation (3) has been conjectured before and is verified to first non-vanishing order in formal expansions of the partition function in $\hbar$ and in a coupling constant for the external vector potentials (Hogreve *et al* 1978). Moreover, it holds in the case of a homogeneous magnetic field $F = \text{constant}$. On the other hand, Avron and Simon have constructed a counterexample based on a Aharonov–Bohm situation where (3) is no longer satisfied. Therefore a quantum mechanical universal paramagnetism of the form (3) cannot be true in general.

As just mentioned, on a formal level the relation (3) has been verified in first non-vanishing order in $\hbar$. This suggests that it may be rigorously valid in the classical limit $\hbar \to 0$. However, for obtaining a reasonable limit of the involved spin variables an appropriate limit procedure is required. Such a procedure has been found and discussed before (Lieb 1973, Gilmore 1979, Simon 1980). Remember that the spin is characterised by a half-integer representation of the rotation group. The key observation for taking the classical limit now is that one has to vary the representation such that in a certain sense it tends to infinity like $\hbar^{-1}$ as $\hbar$ goes to zero. In this way, for example, SO(3) quantum spins approach classical spins which are given by unit vectors on the 2-sphere S^2. The same procedure has been applied to study the classical limit of partition functions for Hamiltonians with Yang–Mills potentials (Hogreve *et al* 1983). In particular, a complete asymptotic expansion has been derived for such quantities by Schrader and Taylor (1984). Below we shall apply these results to spin Hamiltonian operators (1) and obtain for $Z(t;\, H_A)$ a classical partition function of the form

$$Z_{\mathrm{cl}}(t;\, H_A) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}\mu(\rho) \exp\left[-t\left(p^2 - \tfrac{1}{2}\sum_{k,l} F_{kl}(q)\sigma_{kl}(\rho) + V(q)\right)\right].$$

Here the $\sigma_{kl}(\rho)$ are the classical spin variables, i.e. functions on the unit sphere S^2, and $\mathrm{d}\mu$ denotes the normalised measure on S^2.

Furthermore, we shall show that $Z_{\mathrm{cl}}(t;\, H_A)$ is greater than the partition function for vanishing field tensor, namely

$$Z_{\mathrm{cl}}(t;\, H_0) \leqslant Z_{\mathrm{cl}}(t;\, H_A)$$

for all $t > 0$, which is the paramagnetic inequality in the classical limit.

We remark that spinless particles behave quantum mechanically in the opposite way to those with spin. The diamagnetic relation

$$Z(t;\ \tilde{H}_A) \leq Z(t;\ \tilde{H}_0)$$

for spinless Hamiltonians $\tilde{H}_A = \Sigma_j\,(-\mathrm{i}\hbar\nabla_j + A_j) + V$ is equivalent to Kato's inequality for the Laplacian $\Delta_A = (-\mathrm{i}\nabla + A)^2$ and has been proved rigorously by Simon (1979) (cf also Hess *et al* 1977). Since in the classical partition function $Z_{\mathrm{cl}}(t;\ \tilde{H}_A)$ constructed with the classical Hamiltonian $\tilde{H}_A(p, q) = (p + A(q))^2 + V(q)$ by a coordinate translation of the momentum variables the magnetic potentials A drop out, within the classical theory no diamagnetism does exist. This is the content of the so-called 'Bohr–van Leeuven theorem' (Bohr 1911, van Leeuwen 1921).

The classical spin at which we arrive is not unrelated to the classical mechanical spin models used previously. By demanding transitivity of the internal space under group actions, Schulman (1968) has concluded that classical spin models must be described by coset spaces of SO(3); in particular, the space SO(3)/SO(2) of the internal coordinates of the mechanical object which he calls a 'spinning dipole' corresponds to our classical limit manifold S^2. However, our employed classical limit procedure does not lead to the two other possibilities for the group SO(3), namely the coset space consisting of SO(3) itself and the trivial case SO(3)/SO(3). Other classical interpretations and models for spin are discussed in Kochen and Specker (1967) and Bacry (1967).

2. The classical limit

Rather than restricting ourselves to the special case of a three-dimensional Euclidean space, we take as configuration space of our system any smooth N-dimensional Riemannian manifold (M, g_{kl}) which can be compact or non-compact, but which we assume (for simplicity) without boundary. In the non-compact case, it is also assumed to be complete. Before formulating the relevant Hamiltonian operators on this manifold, we briefly recall how they are constructed.

Into the quantum mechanical formalism the spin enters via the transformation properties of the wavefunctions describing the spinning particles. The wavefunctions are required to transform under a representation of the spinor group $\mathrm{Spin}(N, \mathbb{R})$, which is defined $(N \geq 3)$ as the simply connected covering group of $\mathrm{SO}(N, \mathbb{R})$. Hence the fact that the fundamental group of $\mathrm{SO}(N, \mathbb{R})$ is isomorphic to $\mathbb{Z}_2$ implies the exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(N, \mathbb{R}) \to \mathrm{SO}(N, \mathbb{R}) \to 1$. In order to construct a representation of $\mathrm{Spin}(N, \mathbb{R})$ consider a complex Clifford algebra $\mathrm{Cliff}(N)$ generated by the N elements $\gamma_1, \ldots, \gamma_N$, which satisfy the anticommutation relations

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$$

$i, j = 1, \ldots, N$. These relations are invariant under the transformation $\tau_A\gamma_i = \Sigma_{j=1}^N\, a_{ij}\gamma_j$ for any $A = (a_{ij}) \in \mathrm{SO}(N)$. Thus τ_A is an automorphism of $\mathrm{Cliff}(N)$ which can be shown to preserve the centre of $\mathrm{Cliff}(N)$. Since every Clifford algebra is isomorphic to a corresponding matrix algebra and because every automorphism of a matrix algebra is inner, by $\pi(A)\gamma\pi(A)^{-1} = \tau_A\gamma$ a projective representation $A \to \pi(A)$ of $\mathrm{SO}(N)$ is defined. This representation can be made two-valued and becomes a representation of $\mathrm{Spin}(N)$ by considering $\exp(d\pi)$ with $d\pi$ being the representation of the Lie algebra $\mathrm{so}(N)$. From the semisimplicity of $\mathrm{SO}(N)$ the isomorphism $\mathrm{so}(N) \approx \mathrm{spin}(N)$ follows, and for any $A \in \mathrm{SO}(N)$ the representation $d\pi$ can be calculated to be given by

$d\pi(A)=\Sigma_{i<j} a_{ij}\gamma_i\gamma_j=\Sigma_{i<j} a_{ij}\sigma_{ij}$, where $\sigma_{ij}=\frac{1}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)$. Another way to look at this is the following. Each Clifford algebra becomes a Lie algebra by defining for $\beta, \gamma\in \mathrm{Cliff}(N)$ the commutator $[\beta,\gamma]=\beta\gamma-\gamma\beta$ as its Lie operation. Taking the subalgebra $\mathrm{Cliff}_2(N)$ spanned by all the products of two generators of $\mathrm{Cliff}(N)$, i.e. $\mathrm{Cliff}_2(N)=\mathrm{span}_{i<j}\{\gamma_i\gamma_j\}$, the Lie subgroup corresponding to the Lie subalgebra $\mathrm{Cliff}_2(N)$ is then given by $\mathrm{Spin}(N)$.

We want to implement the spin structure for quantum particles moving on the configuration manifold M. To this end let us consider the two principal bundles $\mathscr{F}(M)$ and $\mathscr{S}pin(M)$ over M with structure groups $\mathrm{SO}(N)$ or $\mathrm{Spin}(N)$ respectively. $\mathscr{F}(M)$ denotes the familiar frame bundle, while $\mathscr{S}pin(M)$ is called the 'spin bundle'; its total space is a complex vector space on which $\mathrm{Spin}(N)$ acts irreducibly. Let ϕ be a section of $\mathscr{F}(M)$ and ψ be a section of $\mathscr{S}pin(M)$. For any $x\in M$ we can identify $\phi(x)$ with an element of $\mathrm{SO}(N)$ and $\psi(x)$ with an element of $\mathrm{Spin}(N)$. If for any given ϕ we can find a ψ such that for the canonical projection $s:\mathrm{Spin}(N)\to \mathrm{SO}(N)$ the relation $s(\psi(x))=\phi(x)$ is satisfied for all $x\in M$ then we have defined a spin structure on M. While such a lifting of ϕ is locally always possible, it may not be global, e.g. if the second Stiefel–Whitney class of M is non-zero. In the following we will only consider manifolds admitting a spin structure (for conditions on M sufficient for this see Isham (1978) and Milnor (1963)).

Our next aim is to construct a covariant derivative acting on the wavefunctions which are now sections in a vector bundle associated with $\mathscr{S}pin(M)$. The Riemannian structure of M induces a natural connection on the tangent bundle $T(M)$, namely the Levi-Civita connection. This Levi-Civita connection determines a connection on the frame bundle $\mathscr{F}(M)$ which in turn fixes a connection on the spin bundle $\mathscr{S}pin(M)$. In the same way as the connection on $\mathscr{F}(M)$ is determined by requiring that the corresponding covariant derivatives applied to the N frame have to vanish identically, the connection on $\mathscr{S}pin(M)$ is determined by the vanishing of the spinor derivatives of the γ_k. Here the γ_k, $k=1,\ldots,N$ are generators of a Clifford algebra in each N frame as described above. With respect to local coordinates then the covariant spinor derivative on M is of the form (De Witt 1964)

$$\nabla_j^{\mathrm{s}}=\nabla_j-\tfrac{1}{4}\sum_{k,l}\gamma_k\gamma_l\Gamma_j^{kl} \qquad (4)$$

where ∇_j is the covariant derivative including the Levi-Civita connection and Γ_j^{kl} are the coefficients of the frame bundle connection. Using the antisymmetry $\Gamma_j^{kl}=-\Gamma_j^{lk}$ we can write the 'gauge potentials' as $\Gamma_j=\Sigma_{k,l}\gamma_k\gamma_l\Gamma_j^{kl}=\Sigma_{k,l}\sigma_{kl}\Gamma_j^{kl}$ and (4) becomes $\nabla_j^{\mathrm{s}}=\nabla_j-\frac{1}{4}\Gamma_j$.

Now we are in a position to formulate our Hamiltonians. Let A be a smooth vector potential on M, i.e. a C^∞ section of the cotangent bundle $T^*(M)$. The scalar potential V is assumed to be a continuous real-valued function on M which is bounded below and in the non-compact case sufficiently increasing such that $\int_M \exp(-tV(x))\,\mathrm{dvol}(x)<\infty$ for all $t>0$. Then we define

$$H_A=\left(\sum_{j=1}^{N}\gamma^j(-\mathrm{i}\hbar\nabla_j^{\mathrm{s}}+A_j)\right)^2+V$$

to be the local coordinate expression for the Hamiltonian of a quantum particle with spin on M. Here the γ^j are obtained by raising the indices of the γ_k contracting with the tetrads of the local N frame. Consequently the γ^j obey an anticommutation relation of the form $\gamma^j\gamma^k+\gamma^k\gamma^j=2g^{jk}$, i.e. they generate a Clifford algebra associated with the quadratic form g^{jk}. If the original γ_i are chosen to be Hermitian or antiHermitian, then so are the γ^j and therefore H_A is a symmetric operator on the compactly supported C^∞-sections of a vector bundle associated with $\mathcal{S}pin(M)$. Moreover, with the assumptions above it can be shown that H_A is essentially self-adjoint; the domain of its closure is a space of Sobolev type.

With the help of the Ricci identity for spinors

$$\nabla^s_j\nabla^s_k - \nabla^s_k\nabla^s_j = -\tfrac{1}{4}\sum_{l,m} \gamma_l\gamma_m R^{lm}{}_{jk}$$

the Hamiltonian H_A can be rewritten as

$$H_A = g^{-1/2}\sum_{j,k}(-\mathrm{i}\hbar\nabla^s_j + A_j)g^{jk}g^{1/2}(-\mathrm{i}\hbar\nabla^s_k + A_k) - \mathrm{i}\frac{\hbar}{2}\sum_{j,k}\sigma^{jk}F_{jk} - \frac{\hbar^2}{4}R + V \tag{5}$$

where $g = \det(g_{ij})$, $R^{lm}{}_{jk}$ are the components of the Riemann tensor with the above two indices referring to the local N frame, and where we have used the fact that $\Sigma_{j,k,l,m}\,\gamma^j\gamma^k\gamma_l\gamma_m R^{lm}{}_{jk} = -2R$, R being the curvature scalar. Comparing this with the analogous expression (2) we observe that in the non-flat case the additional term $-\frac{1}{4}\hbar^2 R$ appears which, however, will drop out as $\hbar \to 0$.

Concerning the classical limit of partition functions we can fall back on already existing results. While the limit as $\hbar \to 0$ for partition functions of the above type has been derived by Neumann–Dirichlet bracketing techniques (Hogreve 1983) or via functional integration methods (Hogreve *et al* 1983), Schrader and Taylor (1984) have given a complete asymptotic expansion in $\hbar$. More specifically, they have proved the following: let G be a semisimple connected compact Lie group, $\mathfrak{g}$ its Lie algebra and $P \to M$ a principal G bundle with a given connection on P, the connection being regarded as a $\mathfrak{g}$-valued 1-form. In local coordinates it has the coefficients a_j which are supposed to be C^∞. Let π be an irreducible unitary representation of $\mathfrak{g}$ with fundamental weight λ_0. Recall that the unitary irreducible representations of $\mathfrak{g}$ can be indexed by a lattice in a Weyl chamber, and let π_n denote the representation corresponding to the point $n\lambda_0$ in this chamber. Furthermore, let d_n be the dimension of the representation space of π_n as given by Weyl's formula. Relate $\hbar$ and n by $\hbar = 1/n$. Then as $n \to \infty$ the partition function for the sequence of Hamiltonians

$$H_n = -\hbar^2 g^{-1/2}\sum_{j,k}(\nabla_j + \pi_n(a_j))g^{jk}g^{1/2}(\nabla_k + \pi_n(a_k)) + \mathrm{i}\hbar\pi_n(a_0) + V \tag{6}$$

has a complete asymptotic expansion

$$d_n^{-1}\,\mathrm{Tr}\exp(-tH_n) \sim \hbar^N(Z_0(t) + Z_1(t)\hbar + Z_2(t)\hbar^2 + \ldots) \tag{7}$$

where $Z_0(t)$ is given by the classical partition function

$$Z_{\mathrm{cl}}(t;\,H) = \iint_{T^*(M)}\int_{\Lambda_0} \mathrm{dvol}(x)\,\mathrm{d}^N p\,\mathrm{d}\mu(\lambda)\exp\bigg[-t\bigg(\sum_{j,k} g^{jk}(x)[p_j + \lambda(a_j(x))]$$
$$\times[p_k + \lambda(a_k(x))] + \lambda(a_0(x)) + V(x)\bigg)\bigg].$$

The classical phase space $T^*(M)\times\Lambda_0$ consists of the product of the tangent bundle $T^*(M)$ and the coadjoint orbit $\Lambda_0 \subset \mathfrak{g}^*$ containing Λ_0; $\mathrm{d}\mu$ is the natural normalised measure on Λ_0.

In our case we have $\mathrm{G} = \mathrm{Spin}(N)$ and the irreducible representation of $\mathfrak{g} = \mathrm{spin}(N)$ constructed above is generated by the σ_{ij}. It is easy to verify that the σ_{ij} are unitary;

hence the representation is unitary and by λ_0 we denote the corresponding fundamental weight, and by π_n the representation associated to $n\lambda_0$. The classical limit for (5) follows from the sequence of Hamiltonians

$$H_{A;n}=g^{-1/2}\sum_{j,k}\left(-\mathrm{i}\hbar\nabla_j+\mathrm{i}\frac{\hbar}{4}\pi_n(\Gamma_j)+A_j\right)g^{jk}g^{1/2}\left(-\mathrm{i}\hbar\nabla_k+\mathrm{i}\frac{\hbar}{4}\pi_n(\Gamma_k)+A_k\right)$$
$$-\mathrm{i}\frac{\hbar}{2}\pi_n\left(\sum_{j,k}F_{jk}\sigma^{jk}\right)-\frac{\hbar^2}{4}R+V$$

which are in the form appropriate for the application of (6) and (7). Therefore in the limit as $\hbar=1/n\to 0$

$$\lim_{n\to\infty}\{d_n^{-1}\hbar^{-N}\operatorname{Tr}\exp(-tH_{A;n})\}$$
$$=\iint_{T^*(M)}\int_{\Lambda_0}\mathrm{dvol}(x)\,\mathrm{d}^Np\,\mathrm{d}\mu(\lambda)$$
$$\times\exp\left\{-t\left[\left(\sum_{j,k}g^{jk}(x)(p_j+\tfrac{1}{4}\lambda(\Gamma_j(x))+A_j(x))(p_k+\tfrac{1}{4}\lambda(\Gamma_k(x))+A_k(x))\right.\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\lambda\left(\sum_{j,k}F_{jk}(x)\sigma^{jk}(x)\right)+V(x)\right]\right\}$$
$$=\iint_{T^*(M)}\int_{\Lambda_0}\mathrm{dvol}(x)\,\mathrm{d}^Np\,\mathrm{d}\mu(\lambda)\exp\left\{-t\left[\sum_{j,k}g^{jk}(x)p_jp_k\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\lambda\left(\sum_{j,k}F_{jk}(x)\sigma^{jk}(x)\right)+V(x)\right]\right\} \tag{8}$$

where, by the coordinate shift $P_j\to P_j-\frac{1}{4}\lambda(\Gamma_j(x))-A_j(x)$ in momentum space, the Γ_j and A_j containing terms are eliminated. Only that part of the external electromagnetic field which couples to the spin survives in the classical limit.

3. The classical paramagnetic inequality

We want to prove the classical paramagnetic relation

$$Z_{\mathrm{cl}}(t;H_0)\leqslant Z_{\mathrm{cl}}(t;H_A) \tag{9}$$

for all $t>0$ and smooth vector fields A. To this end we consider the term by which both classical partition functions differ, namely

$$R(t;x)=\int_{\Lambda_0}\mathrm{d}\mu(\lambda)\exp\left[-\tfrac{1}{2}t\lambda\left(\sum_{j,k}F_{jk}(x)\sigma^{jk}(x)\right)\right].$$

Since $\mathrm{d}\mu$ is a probability measure we can apply Jensen's inequality to estimate $R(t;x)$ from below

$$\exp\left[-\tfrac{1}{2}t\int_{\Lambda_0}\mathrm{d}\mu(\lambda)\lambda\left(\sum_{j,k}F_{jk}(x)\sigma^{jk}(x)\right)\right]\leqslant R(t;x) \tag{10}$$

for all $x \in M$, $t > 0$. Next we want to show that the integral over the coadjoint orbit in the exponential of the left-hand side of (10) vanishes. More generally

$$\int_{\Lambda_0} \mathrm{d}\mu(\lambda)\lambda(X) = 0 \tag{11}$$

for any $X \in \mathfrak{g}$, any coadjoint orbit $\Lambda_0 \subset \mathfrak{g}^*$, $\mathfrak{g}$ a semisimple Lie algebra of a compact Lie group G. By definition, the measure μ on Λ_0 is induced by the Haar measure γ on G via $\mu(\Lambda) = d_0\gamma(\Lambda_G)$ for any subset $\Lambda \subset \Lambda_0$ corresponding to a measurable set $\Lambda_G \subset G$; here $\Lambda_G = \{g \in G | \mathrm{Ad}^*(g)\lambda_0 \in \Lambda\}$ and we consider $\lambda_0 \in \mathfrak{h}^*$ (originally defined only on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) as extended to $\mathfrak{g}^*$ by setting it to zero on $\mathfrak{h}^{\perp}$). As before, d_0 denotes the dimension of the representation associated with the fundamental weight λ_0. Observing that, as g ranges over G, $\mathrm{Ad}^*(g)\lambda_0$ ranges over Λ_0 we see that (11) is equivalent to

$$\int_{G} \mathrm{d}\gamma(g)\, \mathrm{Ad}^*(g)\lambda_0(X) = 0. \tag{12}$$

But because Ad* restricted to Λ_0 is irreducible, (12) is an immediate consequence of the Schur orthogonality relation (which can be found, for example, in Kirillov (1976, ch 9.2)). Note that in the case of G being simple, Ad itself is irreducible and the Schur orthogonality gives $\int_G \mathrm{Ad}(g^{-1})\, \mathrm{d}\gamma(g) = 0$ implying (12). Therefore $R(t;x) \leqslant \mathbb{1}$ for all $x \in M$, $t > 0$ and (9) is proved.

Finally, as an explicit example we consider the situation of 'usual' quantum spins, $N = 3$, $G = \mathrm{Spin}(3) \approx \mathrm{SO}(3) \approx \mathrm{SU}(2)$. Here the γ_i can be taken as the familiar Pauli matrices σ_i, $i = 1, 2, 3$, and the coadjoint orbit Λ_0 is isomorphic to the sphere S^2. Since $\sigma_j\sigma_k - \sigma_k\sigma_j = 2\mathrm{i}\sum_{l=1}^{3} \varepsilon_{jkl}\sigma_l$ the term containing the field tensor can be expressed with the help of the dual field tensor $F_j^* = \sum_{k,l} \varepsilon_{jkl}F^{kl}$ as

$$\frac{\hbar}{2\mathrm{i}}\sum_{j,k} F^{jk}\sigma_{jk} = \tfrac{1}{2}\hbar \sum_j F_j^*\sigma_j.$$

Then the classical spin partition function (8) becomes

$$Z_{\mathrm{cl}}(t;H_A) = \int_M \int_{\mathbb{R}^3} \int_{S^2} \exp\left[-t\left(\sum_{j,k} g^{jk}(x)p_jp_k - \tfrac{1}{2}\sum_j F_j^*(x)l_j(\omega) + V(x)\right)\right] \times g(x)^{1/2}\, \mathrm{d}x\, \mathrm{d}^3p\, \mathrm{d}\omega \tag{13}$$

where $\omega = (\theta, \varphi)$ is a vector on S^2 with $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$, $\mathrm{d}\omega = (4\pi)^{-1}\sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ and the l_j can be calculated as $l_1(\omega) = \frac{1}{2}\sin\theta\cos\varphi$, $l_2(\omega) = \frac{1}{2}\sin\theta\sin\varphi$ and $l_3(\omega) = \frac{1}{2}\cos\theta$. Of course (13) can be used to verify the paramagnetic properties of the spin partition function directly. Moreover, choosing special field configurations which allow explicit calculations, (13) shows that the spin part of the classical partition function indeed gives a non-trivial contribution, i.e. in general $Z_{\mathrm{cl}}(t;H_0) < Z_{\mathrm{cl}}(t;H_A)$. So this is a genuine paramagnetic effect.

4. Concluding remarks

Considering general spin Hamiltonians we have seen that they can be written in a form which allows the application of results about the classical limit derived originally within the context of Yang–Mills theory. In the classical limit the spin degrees of freedom manifest themselves as elements of the coadjoint orbit of the spin group (which depend in general on the initially chosen representation). The resulting classical partition function includes integration over such a coadjoint orbit, and by estimates on this integration we have demonstrated that the partition function in an external electromagnetic field is greater than without the field. Thus we have proved the paramagnetism of spin on a classical level. We conclude that non-classical quantum effects must be responsible for the failure of a corresponding general quantum version of the paramagnetic inequality.

Rather than ordinary vector potentials we could have allowed Hamiltonians including non-commutative Yang-Mills potentials A^{YM}. Then simultaneously as $\hbar \to 0$ the representation of the Yang-Mills gauge groups also has to be varied. The classical partition function is still of the form (8), except that now the ordinary field tensor has to be replaced by a classical limit quantity of the Yang-Mills field tensor F^{YM}. Nevertheless, relations (10)-(12) do not depend on these modifications. Therefore also in this situation the paramagnetic inequality (9) holds true.

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